A Causal Algebra for Liouville Exponentials

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Abstract

A causal Poisson bracket algebra for Liouville exponentials on a cylinder is derived using an exchange algebra for free fields describing the *in* and *out* asymptotics. The causal algebra involves an even number of space-time points with a minimum of four. A quantum realisation of the algebra is obtained which preserves causality and the local form of non-equal time brackets.

Following Polyakov's work on the relativistic string [1] the quantum Liouville theory became the subject of intense study. Indeed, the quantisation of Liouville theory is a deep problem in its own right. Early approaches [2, 3, 4, 5, 6] were based on the canonical quantisation of structures present in the classical Liouville theory. In particular, exact forms were obtained for operators corresponding to negative integer and half-integer powers of the Liouville exponential. For arbitrary powers a formal power series can be obtained [6, 7].

In the 1990's attention was focussed on the construction of correlation functions. Dorn and Otto [8] and Zamolodchikov and Zamolodchikov [9] provided formulae for three-point functions and gave a self-consistent framework for computing n-point functions. However, an explicit construction of arbitrary Liouville exponential operators is lacking (see however [10, 11]). In [12] it was noted that the Liouville exponential obeys a causal algebra at the classical and quantum level. The Poisson bracket of Liouville exponentials at different

space-time points can be expressed in terms of the exponential at four points (the two original points plus two more where the light cones emanating from the two points intersect). However, for exponentials defined on a cylinder this four-point relation is only valid if the time separation is sufficiently small. In this Letter a closed causal algebra is derived for all time separations. The number of points depends on the temporal and spatial separations. The construction is based on an exchange algebra for the free fields describing the *in* and *out* asymptotics of the Liouville field. A quantum version of the exchange algebra is obtained through canonical quantisation. This leads to a quantum realisation of the causal algebra for the Liouville exponential to the power $-\frac{1}{2}$.

Classically, the Liouville field, $\varphi(x, \bar{x})$, satisfies the equation of motion

$$\partial_x \partial_{\bar{x}} \varphi(x, \bar{x}) + \mu^2 e^{2\varphi(x,\bar{x})} = 0. \tag{1}$$

Here $x = \tau + \sigma$ and $\bar{x} = \tau - \sigma$ are chiral coordinates (τ is time and σ is the spatial coordinate) and μ is a constant. The Liouville field can be built out of canonical free fields. The simplest object is the exponential in the equation of motion, $e^{2\varphi(x,\bar{x})}$, to the power $-\frac{1}{2}$

$$V(x, \bar{x}) = e^{-\varphi(x,\bar{x})} = E(x, \bar{x}) + F(x, \bar{x}), \tag{2}$$

where E and F are exponentials of massless free fields

$$E(x, \bar{x}) = e^{-\varphi_{in}(x,\bar{x})}, \qquad F(x, \bar{x}) = e^{-\varphi_{out}(x,\bar{x})}. \tag{3}$$

The free fields φ_{in} and φ_{out} match the Liouville field φ in the limits $\tau \to -\infty$ and $\tau \to \infty$, respectively.

We now specialise to a cylindrical spacetime and take the circumference to be 2π so that $\sigma \equiv \sigma + 2\pi$. The free field φ_{in} has a standard Fourier expansion

$$\varphi_{in}(\tau, \sigma) = q + \frac{p\tau}{2\pi} + \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \left(\frac{a_n}{n}\, e^{-inx} + \frac{\bar{a}_n}{n}\, e^{-in\bar{x}} \right), \tag{4}$$

and the non-zero Poisson brackets read

$$\{q, p\} = 1, \qquad \{a_m, a_n\} = \{\bar{a}_m, \bar{a}_n\} = -im\, \delta_{m+n,0}. \tag{5}$$

The Fourier modes of φ_{out} are related to those of φ_{in} through a canonical transformation. This is the classical analogue of the unitary transformation which defines the quantum S matrix. The canonical transformation is most conveniently expressed as a formula for $F(x, \bar{x})$

$$F(x, \bar{x}) = \frac{\mu^2 e^{-\varphi_{in}(x,\bar{x})}}{4 \sinh^2 \frac{1}{2}p} \int_0^{2\pi} dz \int_0^{2\pi} d\bar{z}\; e^{\frac{1}{2}p[\epsilon(x-z)+\epsilon(\bar{x}-\bar{z})]} e^{2\varphi_{in}(z,\bar{z})}, \tag{6}$$

where ϵ denotes the stairstep function defined by

$$\epsilon(x) = 2n+1 \qquad \text{for} \qquad 2\pi n < x < 2\pi(n+1). \tag{7}$$

The field $\varphi(x,\bar{x}) = -\log V(x,\bar{x})$ satisfies the field equation (1) and defines a canonical map exchanging the Liouville and *in* fields [5, 6, 13, 14]. The transformation is not defined for $p = 0$, and to obtain a one to one correspondence the sign of p must be fixed. Our interpretation of (4) as the *in* field is consistent with $p > 0$; if $p > 0$ $F(x,\bar{x}) \to 0$ as $\tau \to -\infty$ so that $V \sim E = e^{-\varphi_{in}}$ in this limit.

A causal algebra for V can be derived through the exchange algebra satisfied by the E and F fields. The classical algebra reads

$$\begin{aligned}
\{E(x,\bar{x}), E(y,\bar{y})\} &= -\frac{1}{4}\, E(x,\bar{x})E(y,\bar{y})\Big(\epsilon(x-y)+\epsilon(\bar{x}-\bar{y})\Big), \\
\{E(x,\bar{x}), F(y,\bar{y})\} &= \frac{1}{4}E(x,\bar{x})F(y,\bar{y})\Big(\epsilon(x-y)+\epsilon(\bar{x}-\bar{y})\Big) \\
&\quad + \frac{1}{2}E(y,\bar{x})F(x,\bar{y})\frac{e^{-\frac{1}{2}p\epsilon(x-y)}}{\sinh\frac{1}{2}p} + \frac{1}{2}E(x,\bar{y})F(y,\bar{x})\frac{e^{-\frac{1}{2}p\epsilon(\bar{x}-\bar{y})}}{\sinh\frac{1}{2}p}, \\
\{F(x,\bar{x}), F(y,\bar{y})\} &= -\frac{1}{4}F(x,\bar{x})F(y,\bar{y})\Big(\epsilon(x-y)+\epsilon(\bar{x}-\bar{y})\Big).
\end{aligned} \tag{8}$$

Using the Fourier expansion (4) the $\{E,E\}$ bracket is easily derived. The derivation of the other two is outlined in the Appendix. The exchange algebra provides the non-equal time Poisson bracket of the Liouville field exponential V in terms of E and F. The result simplifies if $x-y$ and $\bar{x}-\bar{y}$ are in the 'fundamental' domain $(-2\pi, 2\pi)$. This can always be assumed if the temporal separation is less than π. In this case $\epsilon(x-y) = \mathrm{sign}\,(x-y)$ and $e^{-\frac{1}{2}p\epsilon(x-y)} = \cosh\frac{1}{2}p - \sinh\frac{1}{2}p\ \mathrm{sign}(x-y)$ which yields

$$\begin{aligned}
\{V(x,\bar{x}), V(y,\bar{y})\} &= -\frac{1}{4}\Big[E(x,\bar{x})E(y,\bar{y}) + F(x,\bar{x})F(y,\bar{y}) + 2E(x,\bar{y})F(y,\bar{x}) \\
&\quad +2E(y,\bar{x})F(x,\bar{y}) - E(x,\bar{x})F(y,\bar{y}) - E(y,\bar{y})F(x,\bar{x})\Big] \\
&\quad \times\ \Big(\mathrm{sign}(x-y)+\mathrm{sign}(\bar{x}-\bar{y})\Big).
\end{aligned} \tag{9}$$

Causality follows since $\mathrm{sign}(x-y)+\mathrm{sign}(\bar{x}-\bar{y}) = 0$ if $(x-y)(\bar{x}-\bar{y}) < 0$. Using $E(x,\bar{y})E(y,\bar{x}) = E(x,\bar{x})E(y,\bar{y})$ and a similar formula for F, the right hand side of (9) can be expressed in terms of V at the four points $(x,\bar{x})$, $(y,\bar{y})$, $(x,\bar{y})$ and $(y,\bar{x})$ [12]

$$\begin{aligned}
\{V(x,\bar{x}), V(y,\bar{y})\} &= \frac{1}{4}\left[V(x,\bar{x})V(y,\bar{y}) - 2V(x,\bar{y})V(y,\bar{x})\right] \\
&\quad \times\ \Big(\mathrm{sign}(x-y)+\mathrm{sign}(\bar{x}-\bar{y})\Big).
\end{aligned} \tag{10}$$

Outside the fundamental domain the situation is more complicated. Using the identity

$$\frac{e^{-\frac{1}{2}p\epsilon(x)}}{\sinh\frac{1}{2}p} = \sum_{m=-\infty}^{\infty} e^{mp}\Big(1-\mathrm{sign}(x+2\pi m)\Big) \qquad (p>0). \tag{11}$$

and the shift property

$$E(y,\bar{x}-2\pi m)F(x+2\pi m,\bar{y}) = e^{mp}E(y,\bar{x})F(x,\bar{y}), \tag{12}$$

the $\{E,F\}$ bracket in (8) can be rewritten as

$$\begin{aligned}\{E(x,\bar{x}),F(y,\bar{y})\} &= \frac{1}{4}E(x,\bar{x})F(y,\bar{y})\Big(\epsilon(x-y)+\epsilon(\bar{x}-\bar{y})\Big) \\ &\quad + \frac{1}{2}\sum_{n=-\infty}^{\infty} E(y,\bar{x}-2\pi n)F(x+2\pi n,\bar{y})\Big(1-\mathrm{sign}(x-y+2\pi n)\Big) \\ &\quad + \frac{1}{2}\sum_{n=-\infty}^{\infty} E(x+2\pi n,\bar{y})F(y,\bar{x}-2\pi n)\Big(1-\mathrm{sign}(\bar{x}-\bar{y}-2\pi n)\Big).\end{aligned} \tag{13}$$

This leads to the general non-equal time Poisson bracket

$$\begin{aligned}\{V(x,\bar{x}),V(y,\bar{y})\} &= \frac{1}{4}\sum_{n=-\infty}^{\infty}\Big[V(x,\bar{x})V(y,\bar{y})-2V(y,\bar{x}-2\pi n)V(x+2\pi n,\bar{y})\Big] \\ &\qquad \times\Big(\mathrm{sign}(x-y+2\pi n)+\mathrm{sign}(\bar{x}-\bar{y}-2\pi n)\Big).\end{aligned} \tag{14}$$

Note that only a finite number of terms in the sum contribute (in the fundamental domain only the $n=0$ entry); for sufficiently large $|n|$ the points $(x+2\pi n,\bar{x}-2\pi n)$ and $(y,\bar{y})$ have spacelike separation.

The quantum Liouville theory can be accessed through a canonical quantisation of the *in*-field φ_{in} in the Hilbert space $L^2(R_+)\otimes\mathcal{F}$, where $L^2(R_+)$ corresponds to the momentum representation of zero modes and $\mathcal{F}$ is the standard Fock space for the oscillator modes. The brackets (5) are replaced with the commutators

$$[q,p]=i\hbar, \qquad [a_m,a_n]=[\bar{a}_m,\bar{a}_n]=\hbar\, m\, \delta_{m+n,0}. \tag{15}$$

Our aim is to obtain quantum realisations of the exchange algebra (8) and the causal algebra (14). To that end we require operators corresponding to the exponentials E and F. For E one can simply take a normal ordered version of the classical formula (3)

$$E(x,\bar{x}) =: e^{-\varphi_{in}(x,\bar{x})} : . \tag{16}$$

The *in* field representation of F is deformed at the quantum level [6]

$$F(x,\bar{x})=\mu^2\int_0^{2\pi}dz\int_0^{2\pi}d\bar{z}\ :\frac{e^{\frac{1}{2}p[\epsilon(x-z)+\epsilon(\bar{z}-\bar{x})]}}{4(\sinh^2\frac{1}{2}p+\sin^2\frac{1}{4}\hbar)}e^{-\varphi_{in}(x,\bar{x})}e^{2\varphi_{in}(z,\bar{z})}:f(x-z)f(\bar{x}-\bar{z}). \tag{17}$$

Here f is a short-distance factor

$$f(x)=\left(4\sin^2\frac{x}{2}\right)^{\hbar/(4\pi)}. \tag{18}$$

As usual normal ordering is defined by placing creation ($n<0$ modes in (4)) and annihilation operators ($n>0$ modes) to the left and right, respectively. Hermitian normal ordering of the zero modes is assumed [4, 6], i.e. $:e^{2q}g(p):=e^q g(p)e^q$.

A quantum version of the exchange algebra can be derived by computing the operator products $E(x,\bar{x})E(y,\bar{y})$, $F(x,\bar{x})F(y,\bar{y})$, $E(x,\bar{x})F(y,\bar{y})$ and relating these to products of operators defined at different points. The simplest of these is the $E\cdot E$ product which satisfies

$$\frac{E(x,\bar{x})E(y,\bar{y})}{e^{-\frac{1}{4}i\hbar\Theta(x-y,\bar{x}-\bar{y})}}-\frac{E(y,\bar{y})E(x,\bar{x})}{e^{\frac{1}{4}i\hbar\Theta(x-y,\bar{x}-\bar{y})}}=0, \tag{19}$$

where

$$\Theta(x,\bar{x})=\frac{\epsilon(x)+\epsilon(\bar{x})}{2}. \tag{20}$$

This is the quantum version of the $\{E,E\}$ bracket; expanding in powers of $\hbar$ yields $[E(x,\bar{x}),E(y,\bar{y})]=-\frac{1}{2}i\hbar E(x,\bar{x})E(y,\bar{y})\Theta(x-y,\bar{x}-\bar{y})+O(\hbar^2)$. The F operator satisfies the same commutation relation

$$\frac{F(x,\bar{x})F(y,\bar{y})}{e^{-\frac{1}{4}i\hbar\Theta(x-y,\bar{x}-\bar{y})}}-\frac{F(y,\bar{y})F(x,\bar{x})}{e^{\frac{1}{4}i\hbar\Theta(x-y,\bar{x}-\bar{y})}}=0. \tag{21}$$

This follows from the assumption that E and F are related by a unitary transformation, i.e. the existence of the S matrix. It can also be derived directly from (17). The quantum analogue of the $\{E,F\}$ bracket is

$$\begin{aligned}\frac{E(x,\bar{x})F(y,\bar{y})}{e^{\frac{1}{4}i\hbar\Theta(x-y,\bar{x}-\bar{y})}} &- \frac{F(y,\bar{y})E(x,\bar{x})}{e^{-\frac{1}{4}i\hbar\Theta(x-y,\bar{x}-\bar{y})}} \\ &= \frac{i\sin\frac{1}{2}\hbar}{2\sinh\frac{1}{2}p}\left[e^{-\frac{1}{2}p\epsilon(x-y)}\left(\frac{E(y,\bar{x})F(x,\bar{y})}{e^{-\frac{1}{4}i\hbar\Theta(x-y,\bar{y}-\bar{x})}}+\frac{F(x,\bar{y})E(y,\bar{x})}{e^{\frac{1}{4}i\hbar\Theta(x-y,\bar{y}-\bar{x})}}\right)\right.\\ &\qquad\left.+e^{-\frac{1}{2}p\epsilon(\bar{x}-\bar{y})}\left(\frac{E(x,\bar{y})F(y,\bar{x})}{e^{\frac{1}{4}i\hbar\Theta(x-y,\bar{y}-\bar{x})}}+\frac{F(y,\bar{x})E(x,\bar{y})}{e^{-\frac{1}{4}i\hbar\Theta(x-y,\bar{y}-\bar{x})}}\right)\right].\end{aligned} \tag{22}$$

A derivation of this formula is given in the Appendix. Note that p commutes with the product $E\cdot F$. The explicit p-dependence can be removed using (11) and the quantum shift formula

$$\frac{E(y,\bar{x}-2\pi m)F(x+2\pi m,\bar{y})}{e^{-\frac{1}{4}i\hbar\Theta(x-y+2\pi m,\bar{y}-\bar{x}+2\pi m)}}=e^{mp}\frac{E(y,\bar{x})F(x,\bar{y})}{e^{-\frac{1}{4}i\hbar\Theta(x-y,\bar{y}-\bar{x})}}, \tag{23}$$

giving a quantum form of (13). This together with (19) and (21) provides a causal algebra for the operator $V(x,\bar{x}) = E(x,\bar{x}) + F(x,\bar{x})$

$$
\begin{aligned}
&\frac{V(x,\bar{x})V(y,\bar{y})}{e^{\frac{1}{4}i\hbar\Theta(x-y,\bar{x}-\bar{y})}} - \frac{V(y,\bar{y})V(x,\bar{x})}{e^{-\frac{1}{4}i\hbar\Theta(x-y,\bar{x}-\bar{y})}} \\
&\quad = -\frac{i}{2}\sin\frac{\hbar}{2}\sum_{n=-\infty}^{\infty}\left(\frac{V(y,\bar{x}-2\pi n)V(x+2\pi n,\bar{y})}{e^{-\frac{1}{4}i\hbar\Theta(x-y+2\pi n,\bar{y}-\bar{x}+2\pi n)}} + \frac{V(x+2\pi n,\bar{y})V(y,\bar{x}-2\pi n)}{e^{\frac{1}{4}i\hbar\Theta(x-y+2\pi n,\bar{y}-\bar{x}+2\pi n)}}\right) \\
&\qquad\qquad \times\Big(\mathrm{sign}(x-y+2\pi n) + \mathrm{sign}(\bar{x}-\bar{y}-2\pi n)\Big).
\end{aligned}
\tag{24}
$$

If $x-y$ and $\bar{x}-\bar{y}$ are in the interval $(-2\pi, 2\pi)$ only the $n=0$ summand contributes and $\Theta(x-y)$ reduces to $\frac{1}{2}\left(\mathrm{sign}(x-y) + \mathrm{sign}(\bar{x}-\bar{y})\right)$. In this case it is easy to recover

$$
\begin{aligned}
&[V(x,\bar{x}), V(y,\bar{y})] = -i\,\sin\frac{\hbar}{4}\Big(\mathrm{sign}(x-y) + \mathrm{sign}(\bar{x}-\bar{y})\Big)\times \\
&\qquad\left(V(x,\bar{y})V(y,\bar{x}) + V(y,\bar{x})V(x,\bar{y}) - \frac{V(x,\bar{x})V(y,\bar{y}) + V(y,\bar{y})V(x,\bar{x})}{2\cos\frac{1}{4}\hbar}\right).
\end{aligned}
\tag{25}
$$

which was obtained in [12] via Moyal quantisation.

In [8, 9] matrix elements of the form $\langle p|V(x,\bar{x})|p'\rangle$ were interpreted as three-point functions. The causal algebra (24) can similarly be understood as an identity relating four-point functions. The approach followed here should also be applicable to other integrable conformal field theories. The $SL(2,\mathbb{R})/U(1)$ black hole model has a similar free field parameterisation [15, 16] and quantum exchange and causal algebras are expected here as well.

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Appendix

Here we outline the derivation of the classical and quantum exchange algebras. The canonical brackets (5) are equivalent to

$$\{\varphi_{in}(x,\bar{x}),\varphi_{in}(y,\bar{y})\} = -\frac{1}{2}\Theta(x-y,\bar{x}-\bar{y}). \tag{A.1}$$

The $\{E,E\}$ bracket in (8) is an immediate consequence of this formula. To derive the $\{E,F\}$ bracket it helps to write (6) as follows

$$F(x,\bar{x}) = \frac{\mu^2}{4\sinh^2\frac{1}{2}p}E(x,\bar{x})S(x,\bar{x}), \tag{A.2}$$

and

$$S(x,\bar{x}) = \int_0^{2\pi} dz \int_0^{2\pi} d\bar{z}\; e^{\frac{1}{2}p[\epsilon(x-z)+\epsilon(\bar{z}-\bar{x})]}e^{2\varphi_{in}(z,\bar{z})}. \tag{A.3}$$

Accordingly, we require the bracket of E with p and S, the former is

$$\{E(x,\bar{x}),p\} = -E(x,\bar{x}). \tag{A.4}$$

For the $\{E,S\}$ bracket the following rewrite of (A.3) is useful

$$S(x,\bar{x}) = \int_0^{2\pi} dz \int_0^{2\pi} d\bar{z}\; e^{2\varphi_{in}(x+z,\bar{x}+\bar{z})-p}, \tag{A.5}$$

which together with (A.1) yields

$$\begin{aligned}\{E(x,\bar{x}),S(y,\bar{y})\} \;&=\; \frac{1}{2}E(x,\bar{x})\int_0^{2\pi} dz \int_0^{2\pi} d\bar{z}\; e^{2\varphi_{in}(y+z,\bar{y}+\bar{z})-p} \\ &\qquad\qquad \times\left[\epsilon(x-y-z)+\epsilon(\bar{x}-\bar{y}-\bar{z})+2\right].\end{aligned} \tag{A.6}$$

Using the identity

$$\epsilon(x-y-z) = \epsilon(x-y)-\epsilon(z)-\frac{\cosh\frac{1}{2}p}{\sinh\frac{1}{2}p}+\frac{e^{\frac{1}{2}p[\epsilon(x-y-z)-\epsilon(x-y)+\epsilon(z)]}}{\sinh\frac{1}{2}p}, \tag{A.7}$$

(A.5) reduces to

$$\begin{aligned}\{E(x,\bar{x}),S(y,\bar{y})\} \;&=\; E(x,\bar{x})S(y,\bar{y})\left(\Theta(x-y,\bar{x}-\bar{y})-\frac{\cosh\frac{1}{2}p}{\sinh\frac{1}{2}p}\right) \\ &\quad -\;\theta_{-p}(x-y)\,E(x,\bar{x})S(x,\bar{y}) \;\; -\theta_{-p}(\bar{x}-\bar{y})\,E(x,\bar{x})S(y,\bar{x}),\end{aligned} \tag{A.8}$$

where

$$\theta_p(x) = \frac{e^{\frac{1}{2}p\epsilon(x)}}{2\sinh\frac{1}{2}p}. \tag{A.9}$$

This together with (A.4) and the $\{E,E\}$ bracket leads to the formula for $\{E,F\}$. To compute the $\{F,F\}$ bracket one also requires the $\{S,S\}$ bracket.

In the quantum case (19) follows from the commutator

$$[\varphi_{in}(x,\bar{x}),\varphi_{in}(y,\bar{y})] = -\frac{i\hbar}{2}\Theta(x-y,\bar{x}-\bar{y}). \tag{A.10}$$

Equation (22) is less straightforward - this requires the product of the E and F operators. The quantum F defined in equation (17) can be recast in the form

$$F(x,\bar{x}) = \mu^2 c(p) E(x,\bar{x}) S(x,\bar{x}), \tag{A.11}$$

where E is as in (16),

$$c(p) = \frac{1}{4\sinh\frac{1}{2}p\ \sinh\frac{1}{2}(p+i\hbar)}, \tag{A.12}$$

and

$$S(x,\bar{x}) = \int_0^{2\pi} dz \int_0^{2\pi} d\bar{z} : e^{2\varphi_{in}(x+z,\bar{x}+\bar{z})} : e^{-(p-i\hbar)}. \tag{A.13}$$

Note that the shift in p renders S hermitian. Although the product $E(x,\bar{x})S(x,\bar{x})$ involves coincident points it is well defined; the product actually generates the 'short-distance' factors in the manifestly normal ordered form (17). The normal ordering in (A.13) can be avoided at the expense of performing a multiplicative renormalisation of the μ parameter.

As a first step in computing the $E\cdot F$ product we consider the product of $E(x,\bar{x})$ and $S(y,\bar{y})$; using (A.10) the product can be written

$$E(x,\bar{x})S(y,\bar{y}) = \int_0^{2\pi} dz \int_0^{2\pi} d\bar{z}\ : e^{2\varphi_{in}(y+z,\bar{y}+\bar{z})} : e^{-(p-2i\hbar)} E(x,\bar{x}) e^{\frac{1}{2}i\hbar[\epsilon(x-y-z)+\epsilon(\bar{x}-\bar{y}-\bar{z})]}, \tag{A.14}$$

which for $x=y$ and $\bar{x}=\bar{y}$ yields $E(x,\bar{x})S(x,\bar{x}) = S(x,\bar{x})E(x,\bar{x})$. This can be expressed as an exchange relation using the identity (a generalisation of (A.7))

$$\sinh\frac{p}{2}\ e^{\frac{1}{2}i\hbar[\epsilon(x-y-z)-\epsilon(x-y)+\epsilon(z)]} = \sinh\frac{p-i\hbar}{2} + i\sin\frac{\hbar}{2}\ e^{\frac{1}{2}p[\epsilon(x-y-z)-\epsilon(x-y)+\epsilon(z)]}. \tag{A.15}$$

The result reads

$$\begin{aligned} E(x,\bar{x})S(y,\bar{y}) = \frac{\sinh^2\frac{1}{2}p}{\sinh^2\frac{1}{2}(p+i\hbar)}\Big[& -2i\sin\frac{\hbar}{2}\ \theta_{-p}(x-y)\, e^{\frac{1}{2}i\hbar\epsilon(\bar{x}-\bar{y})}\, S(x,\bar{y})E(x,\bar{x}) \\ & -2i\sin\frac{\hbar}{2}\ \theta_{-p}(\bar{x}-\bar{y})\, e^{\frac{1}{2}i\hbar\epsilon(x-y)}\, S(y,\bar{x})E(x,\bar{x}) \\ & -4\sin^2\frac{\hbar}{2}\ \theta_{-p}(x-y)\,\theta_{-p}(\bar{x}-\bar{y})\, S(x,\bar{x})E(x,\bar{x}) \\ & + e^{i\hbar\Theta(x-y,\bar{x}-\bar{y})} S(y,\bar{y})E(x,\bar{x})\Big]. \end{aligned} \tag{A.16}$$

This leads to the operator product

$$
\begin{aligned}
E(x,\bar{x})F(y,\bar{y}) \;=\; v(p)\Big[& -2i\sin\frac{\hbar}{2}\;\theta_{-p}(x-y)e^{\frac{1}{2}i\hbar\epsilon(\bar{x}-\bar{y})}F(x,\bar{y})E(x,\bar{x}) \\
& -2i\sin\frac{\hbar}{2}\;\theta_{-p}(\bar{x}-\bar{y})e^{\frac{1}{2}i\hbar\epsilon(x-y)}F(y,\bar{x})E(x,\bar{x}) \\
& -4\sin^2\frac{\hbar}{2}\;\theta_{-p}(x-y)\;\theta_{-p}(\bar{x}-\bar{y})F(x,\bar{x})E(y,\bar{y}) \\
& +e^{\frac{1}{2}i\hbar\Theta(x-y,\bar{x}-\bar{y})}F(y,\bar{y})E(x,\bar{x})\Big],
\end{aligned}
\tag{A.17}
$$

where

$$
v(p)=\frac{c(p-i\hbar)}{c(p)}\frac{\sinh^2\frac{1}{2}p}{\sinh^2\frac{1}{2}(p+i\hbar)}=\frac{\sinh^2\frac{1}{2}p}{\sinh^2\frac{1}{2}p+\sin^2\frac{1}{2}\hbar}. \tag{A.18}
$$

Equation (A.17) and its hermitian conjugate can be manipulated into (22).

Finally, (A.12) can be used to write the F operator in a more symmetric fashion

$$
F(x,\bar{x})=\frac{\mu^2}{2\sinh\frac{1}{2}p}E(x,\bar{x})S(x,\bar{x})\frac{1}{2\sinh\frac{1}{2}p}. \tag{A.19}
$$

In this form the quantum deformations are hidden in the operator products.

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